Lecture 17: Brownian Motion

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17.1 Brownian Motion as Scaling Limit of Random Walks

Let us start with independent random variables $\xi_1, \xi_2 ...$ that have the same distribution with $\mathbb{E}(\xi_1^2) = 1$, $\mathbb{E}(\xi_1) = 0$. We form the "random walk"

$$S_0 = 0;\ S_n = \sum_{j=1}^{n} \xi_j,\ n \geq 1$$

that these variables generate. This random walk is called simple if $\mathbb{P}(\xi_j = \pm 1) = 1/2$; it captures then the movement of a particle moving on the integer lattice, and kicked to the right, or left, with equal probability, along a discrete and equally spaced time schedule.

Now suppose we want to "zoom out" of this picture, by letting both the size $h > 0$ of the particle's jump, and the unit $\delta > 0$ of time, go to zero; and turn this into a "reel", that is, keep track of it in continuous time:

$$S_0(\omega) = 0; S_t(\omega) = h \sum_{j=1}^{\lfloor t/\delta \rfloor} \xi_j(\omega),\ 0 \leq t < \infty.$$

Of course this "reel", or "movie", is random (the dependence on $\omega \in \Omega$); and depends also on $h > 0, \delta > 0$. How do we send both these parameters to zero, without

(i) throwing the baby out with the bathwater (i.e., getting zero in the limit)

(ii) having the thing explode in our face (i.e., getting ∞ in the limit)?

The answer is in a result we studied last semester, the central limit theorem: take

$$\delta_n = \frac{1}{n},\ h_n = \frac{\sigma}{\sqrt{n}}$$

for some $\sigma > 0$, and let $n \to \infty$.

We get then an entire sequence of random reels

$$S_0^{(n)}(\omega) = 0; S_t^{(n)}(\omega) = \frac{\sigma}{\sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} \xi_j(\omega),\ 0 \le t < \infty,$$

for $n \in \mathbb{N}_0$; and note

(i) For each $t \in (0, \infty)$,

$$S_t^{(n)} \xrightarrow[\mathcal{L}]{n\to\infty} W_t \sim \mathcal{N}(0, \sigma^2 t)$$

by the CLT.

(ii) For arbitrary $m \in \mathbb{N}$, $0 < t_1, .., < t_m < \infty$, the random vector

$$\left(S_{t_1}^{(n)}, S_{t_2}^{(n)} - S_{t_1}^{(n)}, .., S_{t_m}^{(n)} - S_{t_{m-1}}^{(n)}\right)$$

consists of independent variables, and converges in distribution to a vector

$$\left(W_{t_1}, W_{t_2} - W_{t_1}, ..., W_{t_m} - W_{t_{m-1}}\right)$$

of independent variables, each of then $\mathcal{N}(0, \sigma^2(t_j - t_{j-1}))$.

Then, it does not stretch credulity to imagine that the entire sequence $\{S_t^{(n)}, 0 \le t < \infty\}_{n\in\mathbb{N}_0}$ of "random reels", converges in distribution to a random reel - or stochastic process - in a bit more upright parlance, with the following properties; and on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

(i) $\mathbb{P}(W_0 = 0) = 1$.

(ii) For any $0 \le s << \le t < \infty$, the r.v. $W_t - W_s \sim \mathcal{N}(0, \sigma^2(t - s))$.

(iii) The increments $\{W_{t_j} - W_{t_{j-1}}\}_{1\le j\le m}$ are independent.

(iv) $\mathbb{P}\left(\omega \in \Omega :\ \text{the function } t \mapsto W_t(\omega) \text{ is continuous }\right) = 1$.

17.2 Brownian Motion as Random Fourier Series

These properties were formulated by A. Einstein (1905); the last, posits that there is no teleportation of particles, in the limit. He called **Brownian Motion** a stochastic process (family of random variables, random reel) $(W_t)_{0\le t<\infty}$ that satisfies (i)-(iv).

When $\sigma = 1$, we call this process "standard". Unless we mention the opposite, we will be making this choice.

Why do we use the letter W to denote it? To honor N. WIENER (1928), who showed first how to construct this object – from a quite different point of view.

Theorem 17.1 (Brownian Motion as Random Trigonometric Series (N. WIENER, 1928)) *Construct a probability space which enough to support a sequence $Z_0, Z_1, Z_2...$ independent, standard Gaussian random variables. Then the random series*

$$W_t := \frac{1}{\sqrt{\pi}} Z_0(\omega) + \sum_{n \in \mathbb{N}_0} \sum_{k=2^{n-1}}^{2^n - 1} \sqrt{\frac{2}{\pi}} \frac{\sin(kt)}{k} Z_k(\omega),\ 0 \le t \le \pi,$$

converges uniformly for $\mathbb{P}$-a.e., $\omega \in \Omega$; and the resulting family $\{W_t\}_{0 \le t < \infty}$ is standard Brownian motion.

This approach has been refined by LÉVY, CIESIELSKI and, most notably, M. PINSKY (2001). It shows that there exists a r.v. $M : \Omega \to (0, \infty)$ with the property

$$\sup_{0 \le s,t \le 1, t-s \le \delta} |W_t(\omega) - W_s(\omega)| \le M(\omega) \sqrt{\delta \log(\frac{1}{\delta})}$$

for $\mathbb{P}$-a.e., $\omega \in \Omega$.

This comes very close to an amazing result of P.LÉVY (1937), who found the exact modilus of continuity of Brownian motion:

$$g(\delta) = \sqrt{2\delta \log(\frac{1}{\delta})}.$$

We need to develop quite a few more things, before we can place this result in its proper context. But you can certainly go ahead and read its proof on pp.114-116 of BMSC. It only used BOREL-CANTELLI!

$$\mathbb{P}\left(\omega \in \Omega : \limsup_{\delta \downarrow 0} \frac{1}{g(\delta)} \max_{0 \le s < t \le 1, t-s \le \delta} |W_t(\omega) - W_s(\omega)| = 1\right) = 1,$$

the P.LÉVY Modulus of Continuity.

17.3 Gaussian Family

A collection of random variables $\{X_\alpha\}_{\alpha \in I}$ is a **Gaussian family** if the joint distribution

$$X_{\alpha_1}, X_{\alpha_2}, ..., X_{\alpha_n}$$

is multivariate Gaussian for any $n \in \mathbb{N}_0, (\alpha_1, ..., \alpha_n) \in I^n)$.

It is clear that, for such a family, there finite-dimensional distributions are determined entirely, once the means and covariances

$$m_\alpha = \mathbb{E}(X_\alpha), \rho_{\alpha\beta} = \mathbb{E}\left[(X_\alpha - m_\alpha)(X_\beta - m_\beta)\right]$$

have been specified for all $(\alpha, \beta) \in I^2$.

It is also clear that the requirements (ii), (iii) in the definition of Brownian motion, amount to saying that $\{W_t\}_{0 \le t < \infty}$ is a Gaussian family with means

$$m_t = \mathbb{E}(W_t) = 0$$

and covariances

$$\begin{aligned} \rho_{s_t} &:= \mathbb{E}(W_s W_t) = \mathbb{E}\left[W_s\left(W_s + (W_t - W_s)\right)\right] \\ &= \mathbb{E}(W_s^2) + \mathbb{E}\left[W_s(W_t - W_s)r\right] \\ &= s = \min(s, t), \quad \forall 0 < s < t \le \infty. \end{aligned}$$

This observation has important consequences.

17.4 Wiener Measure

The "canonical" space for Brownian motion is $\Omega = C([0, \infty])$, the space of continuous functions

$$\omega : [0, \infty) \to \mathbb{R}.$$

We endow this space with the distance

$$d(\omega_1, \omega_2) := \sum_{n \in \mathbb{N}_0} 2^{-n} \max_{0 \le t \le n} \left(|\omega_1(t) - \omega_2(t)| \wedge 1\right) \tag{17.1}$$

under which it becomes a complete, separable metric space.

We create a measurable space $(\Omega, \mathcal{F})$, by taking $\mathcal{F} = \mathcal{B}(\Omega)$, the σ-algebra generated by the open sets in Ω. This coincides with $\sigma(\mathcal{C})$, the $\sigma-$algebra generated by finite-dimensional "cylinder sets" of the form $\mathcal{C} = \{\omega \ in \Omega : (\omega(t_1), ..., \omega(t_n)) \in A\}; n \in \mathbb{N}_0, A \in \mathcal{B}(\mathbb{R}^n), (t_1, ..., t_n) \in [0, \infty)^n$.

N. WIENER's theorem gives a probability measure on $(\Omega, \mathcal{F})$, denoted by $\mathbb{P}$, under which the coordinate mapping

$$W_t(\omega) = \omega(t), 0 \leq t < \infty$$

is Brownian motion. This is the **WIENER measure**.

More generally, we can start Brownian motion at any arbitrary point $x \in \mathbb{R}$, rather than at the origin. This leads to WIENER measure $\mathbb{P}^x$, with the property

$$\mathbb{P}^x(\omega(0) = x) = 1$$

in addition to properties (ii)-(iv) in the definition of Brownian motion.

For any given $t \in [0, \infty)$, we can consider the collection

$$C_t = \{\omega \in \Omega : (\omega(t_1), ..., \omega(t_n)) \in A\}; n \in \mathbb{N}_0, A \in \mathcal{B}(\mathbb{R}^n), (t_1, ..., t_n) \in [0, t]^n$$

of finite-dimensional cylinder sets, and the smallest σ-algebra $\mathcal{F}_t = \sigma(\mathcal{C}_t)$ of subsets of $\mathcal{F}$ that contains it.

Then it is straightforward to show (Problem 2.4.2) that

$$\mathcal{F}_t = \phi_t^{-1}(\mathcal{F}), \text{ for } (\phi_t \omega)(s) := \omega(t \wedge s), \ 0 \leq s < \infty.$$

We have endowed in this manner the measurable space $(\Omega, \mathcal{F})$ with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t < \infty}$, and with a family $\mathbb{P}^x|_{x \in \mathbb{R}}$ of WIENER measures.

17.5 First Properties

17.5.1 Invariances

WIENER measure is invariant under the operations:

(i) Time Reversal:

$$\tilde{W}_t := W_T - W_{T-t}, \ \ 0 \leq t \leq T \tag{17.2}$$

(ii) Time Inversion:

$$\tilde{W}_t := \begin{cases} tW_{\frac{1}{t}}, & 0 < t < \infty \\ 0, & t = 0 \end{cases} \tag{17.3}$$

(iii) Scaling:

$$\tilde{W}_t := \frac{1}{\sqrt{c}} W_{ct}, \quad 0 \le t < \infty \tag{17.4}$$

(iv) JEULIN-YOR:

$$\tilde{W}_t := W_t \int_0^t \frac{W_s}{s} ds, \quad 0 \le t < \infty \tag{17.5}$$

$$\hat{W}_t := W_t - tW_1 - \int_0^t \frac{sW_1 - W_s}{1 - s} ds, \; 0 \le t \le 1 \tag{17.6}$$

(v) Generalized JEULIN-YOR: With an independent random variable $Z \sim \mathcal{N}(\mu, \sigma^2)$

$$\tilde{W}_t := W_t + tZ - \int_0^t \frac{W_s + sZ + \frac{\mu}{\sigma^2}}{s + \frac{1}{\sigma^2}} ds, \; 0 \le t < \infty \tag{17.7}$$

Let us argue property (ii). Only the continuity of $\tilde{W}$ is here at issue, because both $\{W_t\}_{0<t<\infty}$ and $\{\tilde{W}_t\}_{0<t<\infty}$ are Gaussian families with mean zero and covariance structure $\mathbb{E}(W_t W_s) = \min(t, s)$,$\mathbb{E}(\tilde{W}_t \tilde{W}_s) = ts \min(\frac{1}{t}, \frac{1}{2}) = \min(t, s)$.

For this reason, the event

$$\tilde{F} := \{\lim_{t \downarrow 0} \tilde{W}_t = 0\} = \bigcap_{n \in \mathbb{N}_0} \bigcup_{m \in \mathbb{N}} \bigcap_{q \in \mathbb{Q}, 0 < q \le m} \{|\tilde{W}_q \le \frac{1}{n}|\}$$

has the same probability as the

$$\bigcap_{n \in \mathbb{N}_0} \bigcup_{m \in \mathbb{N}} \bigcap_{q \in \mathbb{Q}, 0 < q \le m} \{|W_q \le \frac{1}{n}| = \{\lim_{t \downarrow 0} W_t = 0\} =: F, \}$$

namely, $\mathbb{P}(F) = \mathbb{P}(\tilde{F}) = 1$. The last equality comes from the fact that W is a Brownian motion.

17.5.2 Strong Law of Large Numbers

$$\mathbb{P}\left(\omega \in \Omega : \lim_{t \to \infty} \frac{W_t(\omega)}{t} = 0\right) = 1. \tag{17.8}$$

Proof: From property (ii) with $\theta = \frac{1}{t}$, we have

$$\frac{W_\theta(\omega)}{\theta} = tW_{\frac{1}{t}}(\omega) = \tilde{W}_t(\omega) \xrightarrow{\theta\uparrow\infty} 0, \text{ for } \mathbb{P} - a.e.\omega.$$

■

17.5.3 The Markov Property

For any $t \in [0, \infty)$ the process

$$B_n := W_{t+u} - W_t, 0 \leq u < \infty$$

is a Brownian motion, and independent of

$$\mathcal{F}_t := \sigma(W_s, 0 \leq s \leq t),$$

the "history of W up to and including t".

We express this by saying that the Brownian motion "forget its past and starts afresh", at any fixed time s. We shall see that this is true also at stopping times.

17.5.4 The Martingale property

Denote by M_t any one of

$$W_t, W_t^2 - t, \exp(\lambda W_t - \frac{\lambda^2}{2}t)$$

for $\lambda \in \mathbb{R}$. Then

$$\mathbb{E}\left(M_t|\mathcal{F}_s\right) = M_s, \mathbb{P} - a.e., \tag{17.9}$$

for $0 \leq s < t \leq \infty$.

We will develop soon an entire "conveyor belt" for producing such Brownian martingales. To get a foretaste, look at the Hermite Polynomials

$$H_n(t,x) = \frac{\partial^n}{\partial\lambda^n}\exp(\lambda x - \frac{\lambda^2}{2}t)|_{\lambda=0} \tag{17.10}$$

(e.g., $H_0(t,x) = 1$, $H_1(t,x) = x$, $H_2(t,x) = x^2 - t$, $H_3(t,x) = x^3 - 3xt$, $H_4(t,x) = x^4 - 6tx^3 + 3t^2$,...). For these

$$M_t^{(n)} = H_n(t, W_t), 0 \leq t < \infty$$

is a martingale for each $n \in \mathbb{N}_0$.

17.5.5 Infinitesimnal Generator; Semigroup

For any bounded, measurable $f : \mathbb{R} \to \mathbb{R}$, the Markov property gives

$$\begin{aligned}\mathbb{E}[f(W_{t+u})|\mathcal{F}_t] &= \mathbb{E}[f(W_t + B_u)|\mathcal{F}_t] \\ &= \mathbb{E}[f(x + B_t)]|_{x=W_t} \\ &= (\Pi_u f)(W_t), \quad 0 \leq t, u < \infty\end{aligned}$$

where $\{\Pi_u\}_{u \geq 0}$ is the **Brownian transition semigroup**

$$(\Pi_u f)(x) := \int_{\mathbb{R}} p_u(x,y) f(y) dy = \mathbb{E}[f(x + B_n)]. \tag{17.11}$$

The semigroup property $\Pi_{t+u} = \Pi_t \cdot \Pi_u = \Pi_u \cdot \Pi_t$ follows from FUBINI-Tonelli and the Chapman-Kolmogorov equations

$$p_{t+u} = \int_{\mathbb{R}} p_t(x,y) p_u(y,z) dy = \int_{\mathbb{R}} p_u(x,y) p_t(y,z) dy \tag{17.12}$$

for the fundamental Gaussian kernel

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}. \tag{17.13}$$

It is a straightforward Exercise that the **Infinitesimal Generator**

$$\mathscr{G} f = \lim_{t \downarrow 0} \frac{1}{t} (\Pi_t f - f) \tag{17.14}$$

of the Brownian transition semigroup, is given by

$$\mathscr{G} f = \frac{1}{2} f"$$

at least for $f \in C_b^2(\mathbb{R})$.

H is also easy to check from this considerations, that the function

$$v(t,x) := (\Pi_t f)(x) = \mathbb{E} f(x + W_t)$$

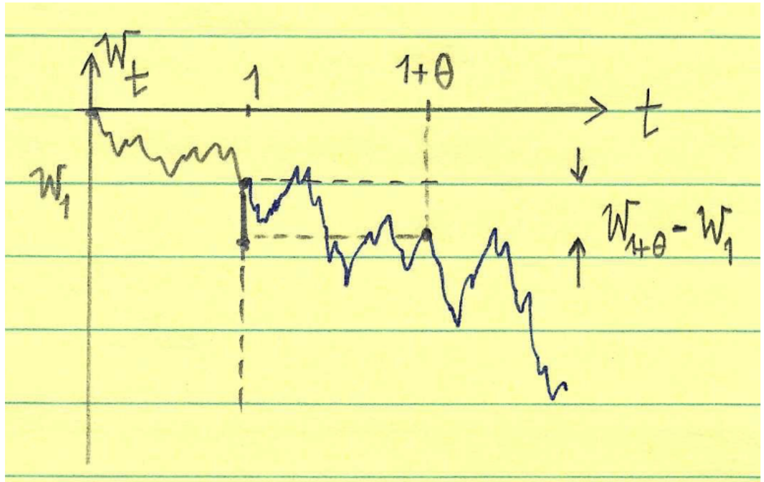

Figure 17.1: Brownian Path

solves then the **Heat equation**

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2}, (t,x) \in (0,\infty) \times \mathbb{R}.$$

17.5.6 Unboundedness and Recurrence

For $\mathbb{P}-$a.e. $\omega \in \Omega$, we have

$$\sup_{0 \le t < \infty} W_t(\omega) = \sup_{0 \le t < \infty} (-W_t(\omega)) = \infty$$

and thus, the set $\{t \ge 0 : W_t(\omega) = b\}$ is unbounded, for any given $b \in \mathbb{R}$.

Thus, Brownian motion is **recurrent**: it visits every site on the real line, and keeps returning to it over and over.

Proof: For the elementary property (iii) of Brownian motion, scaling, the random variable

$$M := \sup_{0 \le t < \infty} W_t$$

has the same distribution as CM, for every $c \in (0,\infty)$.

Thus, the distribution of the random variable M is concentrated on $\{0,\infty\}$:

$$\mathbb{P}(M = 0) + \mathbb{P}(M = \infty) = 1.$$

We need to argue $\mathbb{P}(M = 0) = 0$.

$$\begin{aligned}\mathbb{P}(M=0) &\le \mathbb{P}(W_1 \le 0 \text{ and } W_s \le 0, \forall s \ge 1)\\ &= \mathbb{P}(W_1 \le 0 \text{ and } \sup_{0\le\theta<\infty} (W_{1+\theta} - W_1) \le -W_1)\\ &\le= \mathbb{P}(W_1 \le 0 \text{ and } \sup_{0\le\theta<\infty} (W_{1+\theta} - W_1) \le 0)\end{aligned}$$

This last equation is because $\tilde{W}_\theta = W_{1+\theta}, 0 \le \theta < \infty$ is Brownian motion, and so $\tilde{M} = \sup_{0\le\theta<\infty} \tilde{W}_\theta$ has the same distribution as M; in particular, takes only the values 0 or $+\infty$, so if it is finite it must be zero.

All this comes from the Markov property, which also says that $\tilde{M}$ is independent of W_1. We obtain

$$0 \le p := \mathbb{P}(M=0) \le \mathbb{P}(W_1 \le 0)\mathbb{P}(\tilde{M}=0) = \frac{1}{2}p,$$

thus $p = 0$. ■

17.5.7 The Length of the Brownian Curve $\{W_s, 0 \le s \le t\}$ is Finite

Consider the dyadic rational partition

$$t_j^{(n)} = \frac{j}{2^n}t, j = 0, 1, ..., 2^n$$

of the interval $[0, t]$. We have for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\lim_{n\to\infty} \sum_{j=1}^{2^n} |W(t_j^{(n)}, \omega) - W(t_{j-1}^{(n)}, \omega)|^p = \begin{cases} \infty, & 0 < p < 2 \\ t, & p = 2 \\ 0, & p > 2. \end{cases} \tag{17.15}$$

Proof: Consider the random variables

$$\begin{aligned} D_n &:= \sum_{j=1}^{2^n} \left(W(t_j^{(n)}) - W(t_{j-1}^{(n)})\right)^2 - t = \sum_{j=1}^{2^n} \left[\left(W(t_j^{(n)}) - W(t_{j-1}^{(n)})\right)^2 - \left(t_j^{(n)} - t_{j-1}^{(n)}\right)\right] \\ &= \sum_{j=1}^{2^n} \left(t_j^{(n)} - t_{j-1}^{(n)}\right) \left[\underbrace{\left(\frac{W(t_j^{(n)}) - W(t_{j-1}^{(n)})}{\sqrt{t_j^{(n)} - t_{j-1}^{(n)}}}\right)^2}_{=:Z_j^{(n)}\text{: independent copies of } Z\sim\mathcal{N}(0,1)} - 1 \right] = t2^{-n} \sum_{j=1}^{2^n} \left((Z_j^{(n)})^2 - 1\right) \end{aligned}$$

$\mathbb{E}(D_n^2) = \frac{t^2}{4^n} 2^n \mathbb{E}(Z^2-1)^2 = \text{Const}\frac{t^2}{2^n}, \sum_n \mathbb{E}(D_n^2) < \infty$. Therefore, the sequence $\{D_n^2\}_{n\in\mathbb{N}_0}$ converge to zero **fast** in $\mathbb{L}^2$, thus also in probability. But as we have seen, this implies $\mathbb{P}(\omega \in \Omega : \lim_{n\to\infty} D_n(\omega) = 0) = 1$, which is the first claim.

The second claim for $p = 1$, follows from the inequality

$$t + D_n(\omega) \leq \max_{1\leq j\leq 2^n} |W(t_j^{(n)}, \omega) - W(t_{j-1}^{(n)}, \omega)| \cdot \sum_{j=1}^{2^n} |W(t_j^{(n)}, \omega) - W(t_{j-1}^{(n)}, \omega)|$$

and the continuity of the Brownian path $t \mapsto W_t(\omega)$.

The other claims follow similarly. ■

Coming Attractions: We shall see that all martingale with continuous paths, have similar properties.

17.5.8 Hölder Continuity and Nowhere Differentiability

For a function $f : [0,\infty) \to \mathbb{R}$, let us define

$$D^{\pm} f(t) := \limsup_{h\to 0^{\pm}} \frac{f(t+h) - f(t)}{h} : \text{ upper right (left) derivatives}$$

$$D_{\pm} f(t) := \liminf_{h\to 0^{\pm}} \frac{f(t+h) - f(t)}{h} : \text{ lower right (left) derivatives}$$

Exercise: For every fixed $t \in [0,\infty)$,

$$\mathbb{P}\left[\omega \in \Omega : D^+ W_t(\omega) = -D_+ W_t(\omega) = \infty\right] = 1.$$

In particular, the generic Brownian path is not differentiable at any given, fixed time t.

The following celebrated result, says that if you fix an arbitrary $\omega \in \Omega^*$ in a set $\Omega^* \in \mathcal{G}$ of full Wiener measure, you are not going to be able to find any time $t = t(\omega)$, at which the Brownian path is differentiable.

Theorem 17.2 (Paley, Wiener and Zygmund (1933)) *For $\mathbb{P}$-a.e. $\omega \in \Omega$, the Brownian path $t \mapsto W_t(\omega)$ is nowhere differentiable. More precisely, the set*

$$\{\omega \in \Omega : \text{ for each } t \in [0,\infty), \text{ either } D^+ W_t(\omega) = \infty \text{ or } D_+ W_t(\omega) = -\infty\}$$

contains an event $F \in \mathcal{G}$ with $\mathbb{P}(F) = 1$.

Remark 1 *I don't know whether this set belongs to the σ-algebra $\mathcal{G}$.*

Remark 2 *The "generic" Brownian path has plenty of local maxima (and minima): in fact, the set of such points is countable and dense in $[0, \infty)$. (Theorem 2.9.12 in BMSC).*

At any point t of local maximum, $D^+W_t(\omega) \leq 0$; At any point s of local minimum, $D_-W_s(\omega) \leq 0$.

Thus, in the statement of PWZ, the word "or" cannot be replaced by "and".

the proof of this result is somewhat technical, but not hard. It can even be tweaked a bit, to show that for $\mathbb{P}$-a.e. $\omega \in \Omega$, the Brownian path $t \mapsto W_t(\omega)$ is nowhere HÖLDER continuous with exponent $\gamma > 1/2$. (i.e., not just $\gamma = 1$, as in LIPSCHITZ-continuous).

Remark 3 *We say that a function $f : [0, \infty) \to \mathbb{R}$ is HÖLDER continuous with exponent $\gamma \in (0, 1]$ at $t \in [0, \infty)$, if there exists an open neighbourhood $\mathcal{N}_t \ni t$ and a constant $K_t \in (0, \infty)$,such that*

$$|f(u) - f(s)| \leq K_t |u - s|^\gamma, \forall (u, s) \in \mathcal{N}_t^2.$$

In a similar spirit, we have the following major result (Theorem 2.2.8, pp .53-55).

Theorem 17.3 (KOLMOGOROV-ČENTSOV Theorem) *Suppose that on some given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, there exists a stochastic process $X = \{X_t, 0 \leq t \leq T\}$ with the property*

$$\mathbb{E}|X_t - X_s|^\alpha \leq C|t - s|^{1+\beta}, 0 \leq s, t \leq T$$

for some positive constants α, β, C. Then on this same space there exists a modification $\mathcal{Z} = \{Z_t, 0 \leq t \leq T\}$ of X (meaning, $\mathbb{P}(X_t = Z_t) = 1, \forall t \in [0, T]$), which has a continuous path, in fact is locally HÖLDER continuous with exponent γ, for every $0 < \gamma < \beta/\alpha$:

$$\mathbb{P}\left[\omega \in \Omega : \sup_{0 \leq t, s \leq T, |t-s| \leq h(\omega)} \frac{|Z_t(\omega) - Z_s(\omega)|}{|t - s|^\gamma} \leq D_T\right] = 1.$$

Here $h : \Omega \to (0, \infty)$ is a positive random variable, and $D_T > 0$ an appropriate reak constant.

Again, this proof is technical; but does not rely on anything more fancy than the ČEBYŠEV inequality.

Now, for Brownian motion we have

$$E|W_t - W_s|^{2n} = |t - s|^n \mathbb{E}(Z^{2n}) = C_n |t - s|^n, \forall n \in \mathbb{N}_0.$$

Therefore the conditions are satisfied with $\alpha = 2n, \beta = n - 1$ and we get local HÖLDER continuity for $0 < \gamma < \frac{1}{2} - \frac{1}{n}$, for every $n \in \mathbb{N}_0$. We deduce: **Brownian motion is locally HÖLDER continuity for every exponent** $\gamma \in (0, \frac{1}{2})$.

Discussion: We say that a function $g(\cdot)$ is a modulus of continuity for a given $f : [0, T] \to \mathbb{R}$, if $|f(t) - f(s)| \leq g(\delta)$ holds for $0 \leq s < t \leq T$ with $t - s \leq \delta$. Because of the Law of the Iterated Logarithm, such a modulus for B.M. must be at least as large as $\sqrt{2\delta \log \log(1/\delta)}$; but because of the above, it need not be larger than a constant multiple of δ^γ, for any $\gamma \in (0, 1/2)$.

P.LÉVY's result, states that with

$$g(\delta) = \sqrt{2\delta \log(q/\delta)},$$

$cg(\delta)$ is a modulus of continuity for $\mathbb{P}-$a.every Brownian path on $[0, 1]$, if $c > 1$; but is a modulus of continuity for $\mathbb{P}-$a.no Brownian path on $[0, 1]$, if $0 < c < 1$.

We say that $g(\cdot)$ is the **exact** modulus of continuity.